Resit exam probability probability theory (WIKR-06)

9 July 2020, 8.30 – 12.00

- Work on this exam if the **5th digit** of your student id is **odd**.
- Before the start of the exam, everybody taking part in the exam must sign the student declaration in the exam environment.
- To check for possible fraud, an unannounced sample of students will be contacted soon after the exam.
- The answers need to be written by hand, scanned and submitted within the time limit. You must upload your exam in a single pdf file.
- Every exercise needs to be handed in on a separate sheet.
- Write your name and student number **on every sheet**.
- It is forbidden to communicate with other persons during the exam, except with the course instructor.
- The only tools and aids that you are allowed to use are a non-programmable calculator (not a phone!), and the following material from the nestor course environment:
  a) The pdf file of the lecture notes (not videos, not scribbles).
  b) The pdf files of the tutorial problems.
  c) The pdf files of the homework problems.
  d) The pdf files of the solutions to the homework problems.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes or homework you are using.
- Simplify your final answers as much as possible.
- **NOTA BENE.** Using separate sheets for the different exercises, solving the exam corresponding to your student number, writing your name and student number on all sheets, and submitting all sheets in a single pdf is worth 10 out of the 100 points.

Problem 1 (a:4, b:6, c:8, d:4, e:8 pts).
The joint pdf of a random vector (X,Y) equals

$$f_{X,Y}(x,y) := e^{-c|8x-y|-c|8x+y|}.$$

a) Determine c.

b) Determine the joint pdf $f_{U,V}(u,v)$ of $U := 16X + 2Y$ and $V := 16X - 2Y$. Are U, V independent?

c) Determine the marginal pdf's $f_X(x)$ and $f_Y(y)$. Are X, Y independent? Are X,Y uncorrelated?

d) Determine and simplify $f_{Y|X}(y|x)$. The final result must not contain expressions of the form $|\cdot|$.

e) Let Z be a random variable with pdf $f_Z(z) := f_{Y|X}(z|8)$. Compute $\mathbb{P}(|Z| \leqslant 1 \,|\, Z \in [-64,64])$ and $\mathbb{P}(Z > 400 \,|\, Z > 200)$.

Note. If you could not solve part d), then you may assume (incorrectly) for part e) that $f_{Y|X}(y|x) = \frac{1}{32x}1_{[-8x,8x]}(y) + \frac{1}{2}e^{-(y-8x)}1_{[8x,\infty)}(y)$.

Problem 2 (a:6, b:2, c:6, d:4, e:6, f:6 pts).
A factory produces light bulbs with independent random life times $\{\Delta_n\}_{n\geqslant 1}$. The life time Δ_n of the nth light bulb is a random variable with pdf $f(x) = 4e^{-4x}$.

a) Show that $\mathbb{P}(F) \geqslant \limsup_{n\to\infty} \mathbb{P}\big(\sum_{i\leqslant n}(\Delta_i - 1/4) \geqslant \sqrt{n}\big)$, where

$$F := \Big\{ \sum_{i\leqslant n}(\Delta_i - 1/4) \geqslant \sqrt{n} \text{ for infinitely many } n \geqslant 1 \Big\},$$

and conclude that $\mathbb{P}(F) > 0$.

Set $V_i := 5\exp(-4\Delta_i)$.

b) Show that V_i is uniformly distributed on $[0,5]$.

c) Show that the random variables $\big\{\sqrt[n]{V_1 \cdots V_n}\big\}_{n\geqslant 1}$ converge in probability and determine the limit.

d) For $x \geqslant 0$, let $\lfloor x \rfloor$ be the largest integer smaller than x. Compute the probability that $\lfloor V_1 \rfloor$, $\lfloor V_2 \rfloor$, $\lfloor V_3 \rfloor$ and $\lfloor V_4 \rfloor$ are pairwise distinct.

Let Q be a Poisson random variable with parameter $\lambda = 8$ and set $M := \#\{n \geqslant 1 : \Delta_1 + \cdots + \Delta_n \leqslant 2\}$.

e) Let W be a random variable on $\{0,1,\dots\}$ with pmf $f_W(k) = \frac{4}{5}5^{-k}$, and assume that Q and W are independent. Show that the pmf of $Q+W$ satisfies

$$f_{Q+W}(k) = 4e^{32}5^{-k-1}\mathbb{P}(Z \leqslant k),$$

where Z is a Poisson random variable with parameter 40.

f) Show that $\mathbb{P}(Q=0) = \mathbb{P}(M=0)$ and $\mathbb{P}(Q=1) = \mathbb{P}(M=1)$.

Problem 3 (a:5, b:4, c:2, d:8, e:3, f:8 pts).
Sara the sorceress can perform a magic trick turning snails into beads. There are blue and red snails, as well as blue and red beads. Sara has a big bag containing 200 blue and 800 red snails. A blue snail turns into a blue bead with probability 85% and into a red bead with probability 15%. A red snail turns into a red bead with probability 90% and into a blue bead with probability 10%.

a) Sara draws a snail at random from her bag and performs her trick. What is the probability to craft a blue bead? Assume that the crafted bead is indeed blue. What is the probability that a crafted bead emerged from a blue snail?

Independent of its color, each snail has a random number of speckles, which is uniformly distributed on $\{1,\ldots,400\}$. We now subdivide the 1,000 snails at random into 125 groups consisting of 8 snails each.

b) What is the probability that in at least one of the 125 groups all snails have the same number of speckles?

c) From each of the 125 groups, one representative is selected at random. What is the probability that all 125 representatives have a different number of speckles? You do not need to compute the numerical value of your result.

Assume that among the 1,000 snails there are four *special* ones having precisely 23 speckles.

d) What is the probability that all special snails end up in the same group? How does this answer change if we additionally know that no group contains precisely one special snail?

Sara can undo her trick, reverting the bead into the snail it originated from and putting it back into her bag. Hence, the composition of snails in the bag remains the same. Sara now repeatedly performs and undoes her trick.

e) Let T be the number of trials until crafting a blue bead for the first time. Compute the mgf $M_T(t)$ of T.

f) Suppose that the first crafted bead is red. Compute the expected number of trials until crafting two blue beads in a row.